

From a dissertation submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy in the University of Michigan,
entitled

A STUDY OF PHOTOELASTICITY AND A PHOTOELASTIC AND THEORETICAL
INVESTIGATION OF THE STRESS DISTRIBUTION IN SQUARE BLOCKS
SUBJECTED TO CONCENTRATED DIAGONAL LOADS

By

Max Mark Frocht



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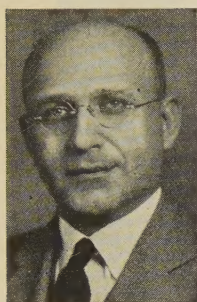


Recent Advances in Photoelasticity

And an Investigation of the Stress Distribution in Square Blocks Subjected to Diagonal Compression

By MAX M. FROCHT,¹ ANN ARBOR, MICH.

After discussing the principle of optical equivalence of stressed isotropic bodies, the author deals respectively with current methods to determine $(P - Q)$; the fringe method; annealing of bakelite; fringe photographs of various models; membrane method to determine $(P + Q)$; the photoelastic investigation of stress distribution in square plates subjected to diagonal compression; isoclinics; trajectories; $(P - Q)$ from fringe method; P and Q by graphical integration. He then summarizes the approximate theoretical analysis by the strain-energy method, gives comparative curves, discusses results, describes new equipment, and in conclusion, derives fundamental equations for the evaluation of P and Q by graphical integration.



THE growth of high-speed and large-size machinery, as well as the personal influence of a few outstanding men, is beginning to convince industrial executives of the need and value of mechanics as an engineering tool. With the general recognition of mechanics must go a special recognition of photoelasticity. Only the simplest problems in stress distribution can be determined theoretically. Generally it is necessary to employ experimental methods to this end, among

which photoelasticity must rank first.

The development of this branch of experimental mechanics has been retarded by the absence of a suitable material and by a certain mistrust among engineers in the validity of its results. The researches of Professors Mesnager, Coker, Filon, and others have gone a long way to dispel these doubts, and to build up a confidence in the results obtained by the optical method. It has been proved theoretically and substantiated experimentally that within the elastic limit and in simply connected bodies the stress distribution is independent of the physical constants

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NOTE: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

of the material. Professor Filon has shown that even in the case of multiply connected bodies such as connecting rods, the errors due to the difference between the materials of the model and the corresponding machine part or structural member are very small. The introduction of celluloid as a working material made photoelastic research not only possible but practicable.

In very recent years interesting and promising developments have taken place, and it is the object of this paper to discuss some of these developments and their application. In particular the author will present, first, Dr. Tuzi's fringe method for the determination of the difference between the principal stresses at a point; second, a new experimental method for the determination of the sum of the principal stresses; third, a process for the annealing of bakelite which will restore this excellent material to photoelastic work; fourth, a photoelastic and theoretical investigation of the stress distribution in square plates subjected to diagonal compression; and fifth, photographs of new and inexpensive laboratory equipment which will demonstrate that a photoelastic laboratory is financially within the reach of industrial organizations and engineering schools.

ESSENTIALS OF PHOTOELASTICITY

The fundamental optical phenomenon underlying all photoelastic investigations is that of double refraction. It explains what is transpiring in the crossed Nicols, the quarter-plates, as well as in the stressed model itself.

When a ray of monochromatic light enters a doubly refracting crystal, such as Iceland spar, it emerges transformed in two important respects: first, it splits into two plane polarized rays, the ordinary and extraordinary, which vibrate in mutually perpendicular planes; second, the two rays travel through the crystal with different velocities, and therefore emerge from the crystal with a phase difference which depends on the wave length of the light and the length of the crystal. It is an experimental fact that a stressed element in an isotropic body behaves just like a doubly refracting crystal. To generalize: *Within the elastic limit all isotropic bodies subjected to a two-dimensional stress system are optically equivalent to a set of a large number of doubly refracting crystals shaped so that their lengths at every point are directly proportional to the difference between the principal stresses at that point, and whose principal planes are set parallel to the stress trajectories.* It is a further experimental fact that the relative retardation R between the ordinary and extraordinary rays produced at a point of an isotropic body due to its temporary doubly refracting character is given by the equation

$$R = c (P - Q) \dots \dots \dots [1]$$

where c is an optical constant depending upon the material, and P and Q are the principal stresses at the point. Due to the above-mentioned facts a stressed, transparent model in a field of *white, circularly polarized light* gives a colored image each tint of which is a measure of $(P - Q)$. This phenomenon is the foundation of the prevailing methods in photoelastic laboratories today. It is the basic idea in color matching and compensating. Both of these methods depend upon the color effect to determine the numerical value of $(P - Q)$.

THE FRINGE METHOD

If instead of white light a monochromatic source be used, then the relative retardation between the ordinary and extraordinary rays caused by the stressed model manifests itself not in color effects but in a change of intensity. Referring now to the rays emerging from the analyzer, let A = amplitude, A_m = maximum amplitude, I = intensity, and I_m = maximum intensity.

It can then be shown² that $A = A_m \sin \pi R$ [2]

Since the intensity of light varies as the square of the amplitude,

$$I = A^2 = A_m^2 \sin^2 \pi R$$

But

$$A_m^2 = I_m$$

hence

$$I = I_m \sin^2 \pi R$$
..... [3]

where R is given by Equation [1].

Inspection of Equation [3] shows that I will be zero for R equal to an integral number of wave lengths. For such values of R the light will be completely extinguished and a dark spot will be obtained on the screen. In this manner the use of mono-

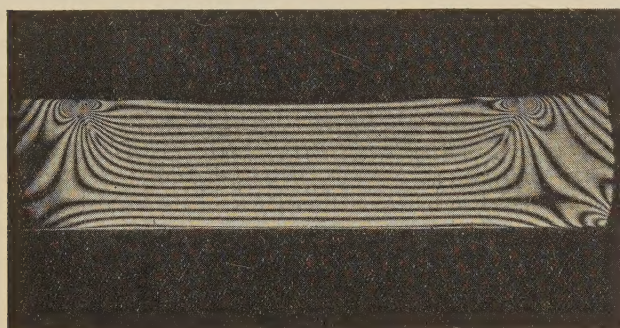


FIG. 1 FRINGE PHOTOGRAPH OF BAKELITE BEAM IN PURE BENDING

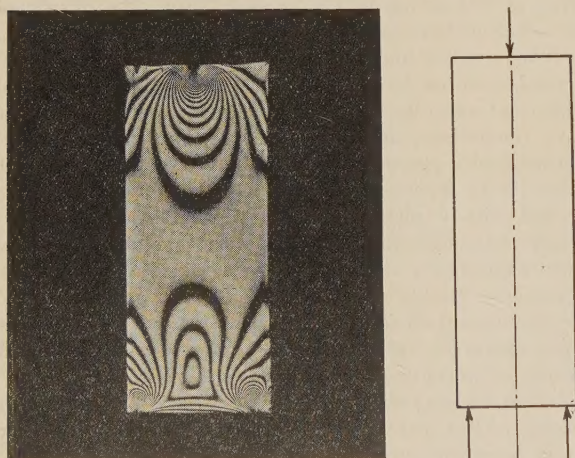


FIG. 2 FRINGE PHOTOGRAPH OF THE ESCAPING TYPE

chromatic light results in alternate bright and black fringes, whose number and complexity depend on the material and loading.

Fig. 1 shows a fringe photograph of a bakelite beam in pure bending. It will be observed that the fringes are straight, parallel, equidistant lines, this being due to the linear distribution of the stress. From the known load conditions and the dimensions of the beam, the stress value of each fringe can be computed.

² See article by A. L. Kimball, Jr., on "Stress Determination by Means of the Coker Photo-Elastic Method," *General Electric Review*, January, 1921, pp. 80-81.

The difference between the principal stresses corresponding to each fringe of a stressed model of the same material becomes thus a known quantity, provided we know the number of wave lengths of relative retardation responsible for the particular fringe, that is, provided we know the *fringe order*.

THE FORMATION OF FRINGES

The formation of fringes is a very interesting, though somewhat complicated phenomenon. Dr. Tuzi classifies fringes into four types: the assembling, the converging, the diverging, and the escaping. The author proposes a simpler classification and will mention only two types of fringes: the escaping and the non-escaping. When all the fringes formed remain in the image, we have the non-escaping type. When some escape from the picture, we have the escaping type. The non-escaping type is best illustrated by pure bending, Fig. 1. The first fringes appear at the extreme fibers and approach the neutral axis as a limit but never reach it, since the stress at the neutral

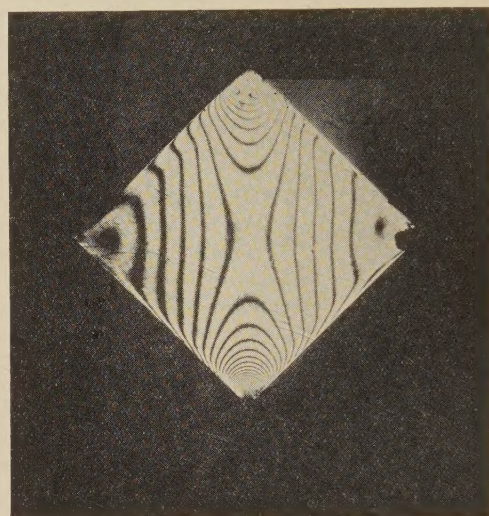


FIG. 3 PHENOLITE MODEL SHOWING EFFECT OF INITIAL STRESSES

axis always remains zero. The fringes crowd together, but do not escape.

The escaping type is the more general of the two. A simple illustration of this type is afforded by the fringes formed in a narrow, rectangular block in compression. Here the fringes are born in the interior, spread toward the edges, and escape, leaving only fragments at the top and bottom. Fringes may escape through the interior as well as the boundary. Fig. 2 shows a fringe photograph of the escaping type.

In the non-escaping type a static picture reveals the fringe order. This is not the case in the escaping type. Here the fringe formation must be studied in order to determine the fringe order. By slow and repeated loading and unloading one can determine the points where the fringes originate and vanish. It is also possible to take moving pictures and thus reproduce the fringe formation on the screen at leisure.

The fringe method for the determination of $(P - Q)$ is simpler than color matching, and more accurate than compensating. The fringes are clearer than isochromatics, and can be traced with accuracy. It eliminates the strain and tediousness that necessarily go with compensating. This method has another important merit. It not only reveals clearly the stresses in the model, but it makes it practicable to photograph them and thereby to secure permanent and convincing evidence of the stress distribution. From a pedagogical point of view the author

considers this of great importance. A picture carries a veracity which a sketch does not, and it is believed that with the help of the fringe method the dissemination of photoelastic knowledge should be materially accelerated.

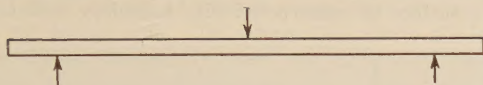
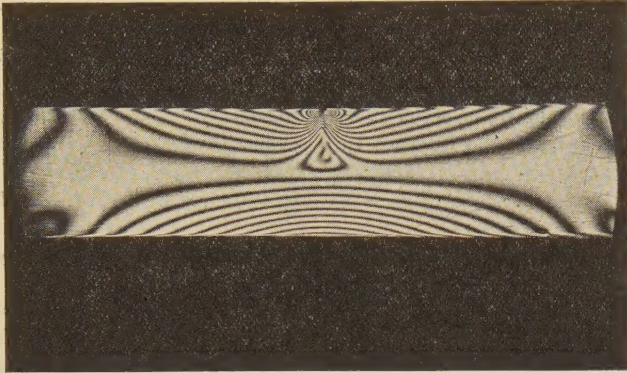


FIG. 4 BAKELITE BEAM UNDER CENTRAL LOAD

ANNEALING OF BAKELITE

The success of the fringe method depends upon having a sensitive material, which will show many fringes under comparatively small loads. Dr. Tuzi used phenolite, which is the invention of Professor Kita and Mr. Matui of the Institute of Physical and Chemical Research, Tokyo, Japan. This material is very sensitive and gives excellent results, but is not obtainable in this country.

Bakelite is a transparent material as sensitive as phenolite, and easily obtainable. Its use as a photoelastic material has been retarded because it shows pronounced initial stresses in its unstrained state. It is therefore of some interest that the author found a method which

proved effective in the annealing of bakelite.³ The main steps of this method are:

1 Heating the models to a temperature of about 70 deg. cent., and

³ Prof. L. N. G. Filon in his paper "On the Graphical Determination of Stresses From Photo-Elastic Observations," *Engineering*, October 19, 1923, states that a method for the annealing of bakelite was discovered by Mr. Jessop. He does not, however, give an account of it, nor does he give any reference to a publication where such an account might be found.

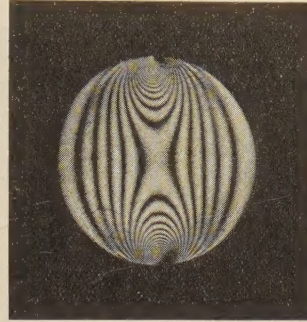


FIG. 7 PHENOLITE DISK

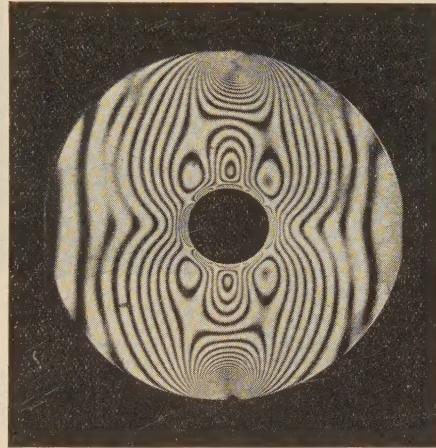
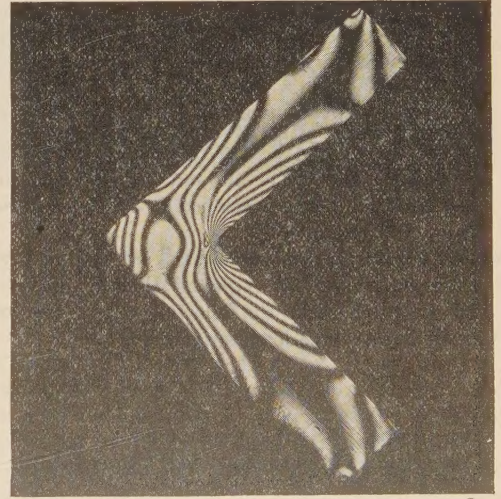
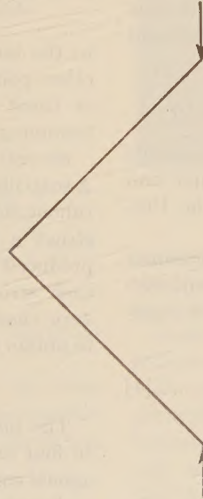


FIG. 8 BAKELITE MODEL



FIGS. 5 AND 6 BAKELITE MODELS SHOWING EFFECT OF FILLET ON STRESS, CONCENTRATION, DIMENSIONS AND LOADS IDENTICAL

2 Cooling them very slowly to room temperature.

To procure uniform heating and cooling it is necessary to avoid direct contact between the models and the walls of the furnace. To this end it is advisable to place the models between glass plates and support them on a block.

The amount of time necessary for cooling depends on the furnace. In the author's case it took sixteen hours. Caution must be exercised not to remove the specimen too soon. Good models have been spoiled by premature removal. The heating can be done rapidly. It took the author only fifteen minutes

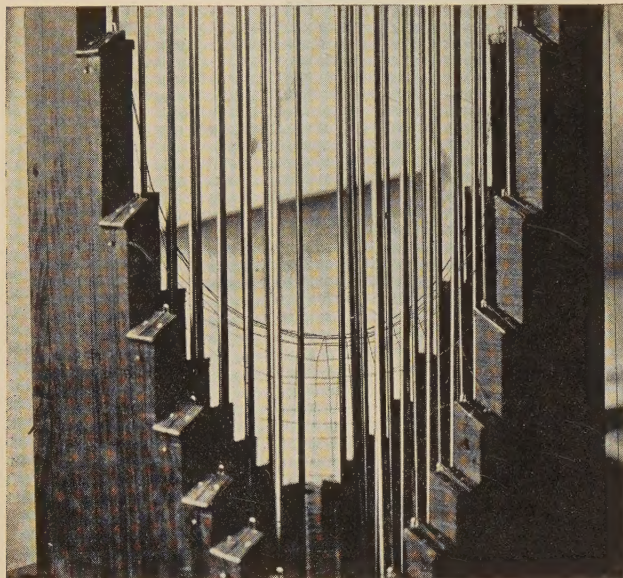


FIG. 9

to do it. In cases of stubborn stresses a second annealing will be found effective. The quality of the annealing obtained from this process can be judged from the photograph reproduced in Fig. 1.

It will be observed that the fringes are straight, parallel, equidistant lines, which would not be the case had there been initial stresses. Fig. 3 shows a phenolite model without annealing. It will be observed that there is marked absence of symmetry in the fringes, and also that they meet the boundary in directions almost parallel to them. An annealed model is shown in Fig. 16 in which the fringes are symmetrical and normal to the boundaries. Figs. 4-8 show fringe photographs of different models annealed by the aforementioned process.

A NEW METHOD FOR THE DETERMINATION OF $(P + Q)$

A method giving the same data as the lateral extensometers of Mesnager and Coker but radically different in principle and construction is being developed in the laboratories of the University of Michigan.

The sum of the principal stresses $(P + Q)$ in a two-dimensional system lends itself to representation by means of a surface.⁴ This can be shown in the following way: The continuity equation of a two-dimensional system is

$$\nabla^4 \phi = 0 \dots \dots \dots [4]$$

where ϕ represents the stress function.

This can be put in the modified form as follows:

⁴ This suggestion is due to Mr. Den Hartog of the Westinghouse Electric and Manufacturing Company, East Pittsburgh, Pa.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \times \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \dots \dots \dots [5]$$

The second factor of this equation represents the sum of the normal component stresses, i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \sigma_x + \sigma_y \dots \dots \dots [6]$$

which sum will be designated by the letter S .

The continuity equation then assumes the form:

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0 \dots \dots \dots [7]$$

and this is the equation of a membrane of small deflections whose boundary is subjected to uniform tension.

At a free boundary one of the principal stresses vanishes, so that the sum of the principal stresses on such boundaries equals the difference, and can therefore be determined photoelastically. If now a surface be constructed with boundary ordinates equal

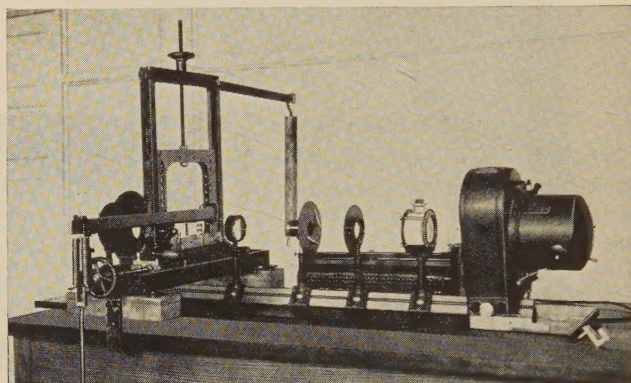


FIG. 10 APPARATUS USED IN PHOTOELASTIC INVESTIGATION

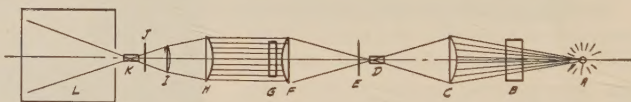


FIG. 11 ARRANGEMENT OF APPARATUS SHOWN IN FIG. 10

A—Light source
B—Water cooler
C—Lens; $F = 100$ mm.
D—Polarizer; 10-mm. Ahrens prism.
E—Quarter-plate
F—Lens; $F = 150$ mm.
G—Test model
H—Lens; $F = 150$ mm. Ahrens prism.
I—Lens
J—Quarter-plate
K—Analyzer; 10-mm. Ahrens prism
L—Camera or screen

to the boundary values of $(P + Q)$, then the ordinates at all other points would represent the sum of the principal stresses at those points. This affords an experimental means of determining $(P + Q)$.

Several methods can be used to construct such a surface. A soap film is one possibility; a membrane formed by a stretched rubber sheet is another. Fig. 9 shows a rough attempt to construct a woven surface. The uniform boundary tension is produced by means of small weights attached to the ends of each string. The shape of the curves thus obtained follows very closely the actual $(P + Q)$ curves. Work is in progress to obtain quantitative results.

THE MAIN PROBLEM

The main problem of which this paper is the outgrowth was to find the stress distribution in a square plate subjected to diagonal compression. The problem was investigated theoretically and photoelastically, and, as will be seen later, good agreement was obtained between the two sets of results. The photoelastic

investigation will be first presented. The apparatus used and its arrangement are shown in Figs. 10 and 11.

The optical solution of a problem consists of four steps:

- 1 Determination of isoclinic lines
- 2 Determination of stress trajectories
- 3 Evaluation of ($P-Q$)
- 4 Evaluation of P and Q separately.

Isoclinics. The locus of points whose principal stresses have the same directions is called an isoclinic line. The simplest and general method to obtain such lines is by removing the quarter-plates from the polariscope and to turn the crossed Nicols 5 or 10 deg. at a time. The image obtained contains the isoclinics for the particular setting of the Nicols. It is of interest that this ingenious method was proposed by Maxwell when he was only nineteen and still a student at Edinburgh.

Glass models, because of their transparency, give the clearest curves, and such models were used in the present investigation. The size of the model does not matter. The glass was $\frac{5}{16}$ in. thick and could sustain an average compressive stress of nearly one ton per square inch. To sustain that stress the load had to be uniformly distributed. This was attained by the use of soft copper pins which also prevented chipping. The original sketch of isoclinic curves was enlarged, and the final curves shown in Fig. 12 represent the averages of the enlargement.

The author wishes to call attention to a possible source of error in sketching isoclinics. Not all the black curves from plane polarized light are necessarily isoclinics. A point of zero stress or a point of equal principal stresses will also cause a dark spot. To eliminate the possibility of error, one should compare the pictures with and without quarter-plates. The effect of the quarter-plates, among other things, is to remove the isoclinics,

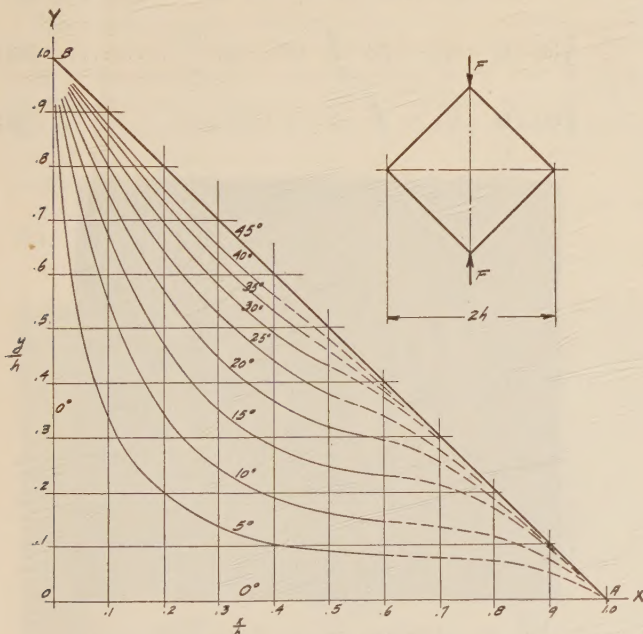


FIG. 12 ISOCLINIC LINES

(Solid lines: from direct observation; broken lines: from theoretical considerations.)

as can be seen from Figs. 13 and 14. The absence of the cross from Fig. 14 shows that it is an isoclinic, and likewise the presence of the circle in the same photograph shows that the circle is not an isoclinic.⁵

⁵ Professor Muira includes the circle among the zero isoclinics, but the author believes this to be an error. See the former's "Spannungskurven in rechteckigen und keilförmigen Trägern," p. 87.

An interesting use was made of the cross representing the zero isoclinic in symmetrical models. During the investigation the Ahrens prisms became loose in their mountings and turned about the horizontal axis. They were reset by crossing the

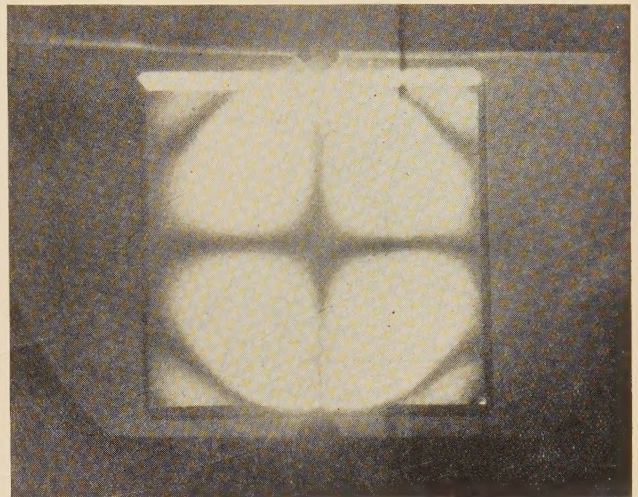


FIG. 13 WITHOUT QUARTER-PLATES

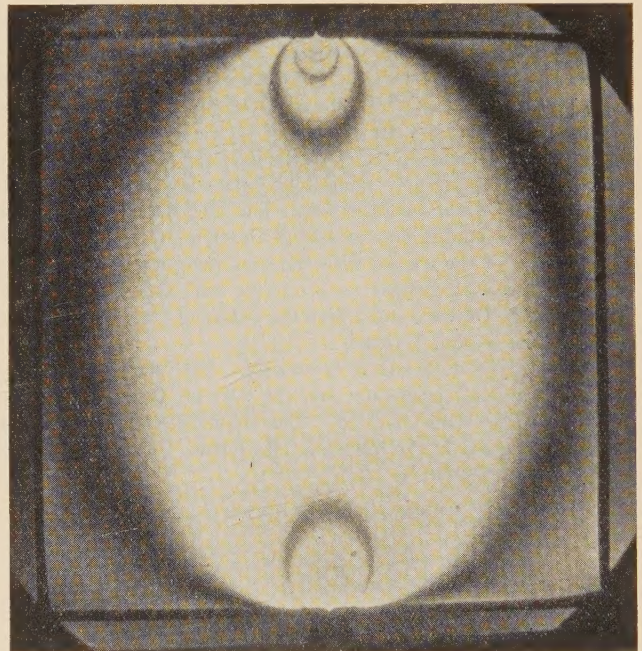


FIG. 14 WITH QUARTER-PLATES

mountings in a vertical and horizontal position, and adjusting the prisms until the isoclinic curve of the symmetrical model gave the cross.

Stress Trajectories. These were mapped in the usual manner and are shown by the full lines of Fig. 15.

Determination of ($P - Q$). The difference between the principal stresses was determined in several ways. Celluloid, bakelite, and phenolite were used for materials, and color matching, tension compensating, and fringe counting as methods. But the author is so impressed with the simplicity and accuracy of the fringe method that the results will be presented in this form. First, two well-annealed models of the same strip of

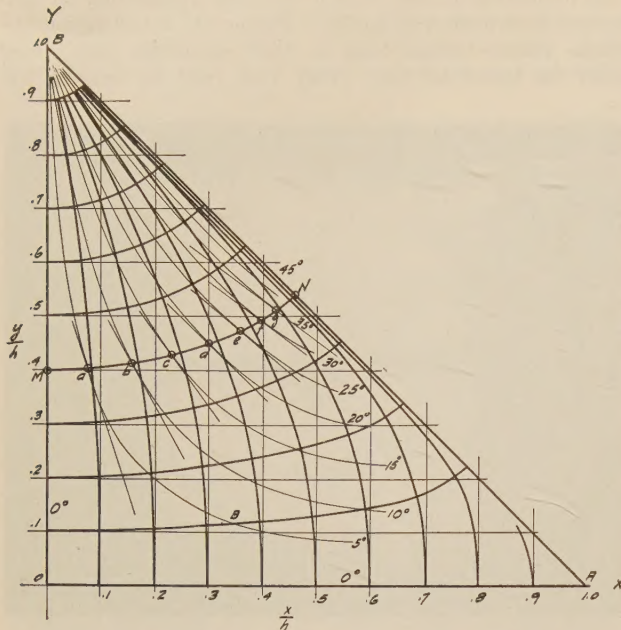


FIG. 15 STRESS TRAJECTORIES (Heavy Lines) AND ISOCLINES (Light Lines)

bakelite were secured: a square plate and a beam, Figs. 16 and 17. The sodium lamp which was used for isoclines was then replaced by the mercury-vapor lamp shown in Fig. 18, and the photographs shown in Figs. 16 and 17 taken. This gave all the optical data necessary to determine $(P - Q)$. As a measure of safety a free-hand sketch of the fringes was made immediately after taking the picture. This took but a few minutes and no optical creep could be observed. The clearness and thinness of the fringes enable one to sketch these with accuracy. The intersections of the fringes with the axes were noted, and curves were plotted with fringes as ordinates and axes as abscissas. Columns 2 in Tables 1 and 2 give the values of $(P - Q)$ in fringe order for the X- and Y-axes.

Determination of P and Q Separately. The principal stresses were determined separately by the use of Professor Filon's graphical-integration method. The equations used in this method are given in the following two forms. For their derivation see Appendix No. 1.

TABLE 1 COMPUTATIONS OF P AND Q ALONG OX

x/h (1)	$(P - Q)$ in fringe order (2)	$(P - Q) \Delta\phi$ $\Delta\phi = 0.0873$ radians (3)	Δy in inches (4)	$(P - Q) \frac{\Delta\phi}{\Delta y}$ (5)	Mean $(P - Q) \frac{\Delta\phi}{\Delta y}$ (6)	$(P - Q) \Delta s_1$ (7)
0.0	4.17	0.3639	∞	0.0	0.120	0.060
0.1	4.00	0.3490	1.690	0.2065	0.268	0.134
0.2	3.58	0.3124	1.000	0.3124	0.364	0.182
0.3	3.00	0.2618	0.710	0.3687	0.382	0.191
0.4	2.25	0.1963	0.510	0.3849	0.344	0.172
0.5	1.47	0.1236	0.440	0.2809	0.260	0.130
0.6	0.88	0.0768	0.430	0.1786	0.132	0.066
0.7	0.45	0.0393	0.370	0.1062	0.064	0.032
0.8	0.20	0.0174	0.330	0.0527	0.028	0.014
0.9	0.05	0.0044	0.240	0.0183	0.007	0.003
1.0	0.0	0.0	0.0	0.0		

x/h (1)	P (8)	Q (9)	P in lb. per sq. in., $f_1 = 283.88$ (10)	Q in lb. per sq. in., $f_1 = 283.88$ (11)	Mean Q in lb. per sq. in. (12)
0.0	0.9845	-3.1855	279.5	-904.3	-895
0.1	0.9245	-3.0755	262.5	-873.1	-837
0.2	0.7905	-2.7895	224.4	-791.9	-735
0.3	0.6085	-2.3915	172.7	-679.0	-596
0.4	0.4175	-1.8325	118.5	-520.2	-428
0.5	0.2455	-1.2245	69.6	-347.6	-277
0.6	0.1155	-0.7645	32.8	-217.0	-160
0.7	0.0495	-0.4005	14.05	-113.7	-75
0.8	0.0175	-0.1825	4.97	-51.8	-25
0.9	0.0035	-0.0465	0.99	-13.2	-3
1.0	0.0	0.0	0.0	0.0	0

TABLE 2 COMPUTATIONS OF P AND Q ALONG OY

y/h (1)	$(P - Q)$ in fringe order (2)	$(P - Q) \Delta\phi$ $\Delta\phi = 0.08726$ radians (3)	Δx in inches (4)	$(P - Q) \frac{\Delta\phi}{\Delta x}$ (5)	Mean $(P - Q) \frac{\Delta\phi}{\Delta x}$ (6)	$(P - Q) \Delta s_2$ (7)
0.0	4.17	0.3639	5.00	0.0728	0.100	0.050
0.10	4.20	0.3665	2.12	0.1728	0.240	0.120
0.20	4.30	0.3752	1.03	0.3642	0.465	0.232
0.30	4.48	0.3909	0.60	0.6515	0.800	0.400
0.40	4.79	0.4180	0.40	1.0450	1.400	0.700
0.50	5.31	0.4634	0.26	1.7820	2.350	1.175
0.60	6.20	0.5410	0.17	3.1820	4.400	2.200
0.70	7.75	0.6763	0.12	5.7800	6.950	1.740
0.75	9.30	0.8130	0.097	8.3500	11.250	2.810
0.76	9.70	0.8460	0.094	9.0000	8.700	0.435
0.77	10.40	0.9080	0.090	10.0300	9.500	0.475
0.78	11.50	1.0030	0.087	11.5400	10.800	0.540
0.79	12.50	1.0900	0.083	13.0600	12.300	0.615
0.80	13.50	1.1790	0.080	14.7500	13.900	0.695

y/h (1)	P in fringe order (8)	Q in fringe order (9)	P in lb. per sq. in., $f_1 = 283.88$ (10)	Q in lb. per sq. in., $f_1 = 283.88$ (11)	Mean P in lb. per sq. in. (12)
0.0	0.9845	-3.1855	279.5	-904.3	277
0.10	0.9645	-3.2355	273.8	-918.5	272
0.20	0.9445	-3.3555	268.0	-952.6	262
0.30	0.8920	-3.5880	253.2	-1018.6	243
0.40	0.8020	-3.9880	227.7	-1132.0	208
0.50	0.6220	-4.6880	176.6	-1330.0	140
0.60	0.3370	-5.8630	95.7	-1664.4	20
0.70	-0.3130	-8.0630	-88.8	-2289.0	-122
0.75	-0.5030	-9.8030	-142.8	-2783.0	-100
0.76	-0.5650	-10.2650	-160.4	-2914.0	(Mean P
0.77	-0.3400	-10.7400	-96.5	-3050.0	between
0.78	0.2200	-11.2800	62.4	-3202.0	0.75
0.79	0.6050	-11.8950	171.7	-3377.0	and
0.80	0.9100	-12.5900	258.0	-3574.0	0.80

is 150)

$$1 \quad \begin{cases} (a) & P = P_0 - \Delta\phi \int (P - Q) \frac{ds_1}{\Delta y} \dots \dots \dots [8a] \\ (b) & P = P_0 + \int (P - Q) \cot \psi d\phi \dots \dots \dots [8b] \end{cases}$$

$$2 \quad \begin{cases} (a) & Q = Q_0 - \Delta\phi \int (P - Q) \frac{ds_2}{\Delta x} \dots \dots \dots [9a] \\ (b) & Q = Q_0 + \int (Q - P) \cot \psi_1 d\phi \dots \dots \dots [9b] \end{cases}$$

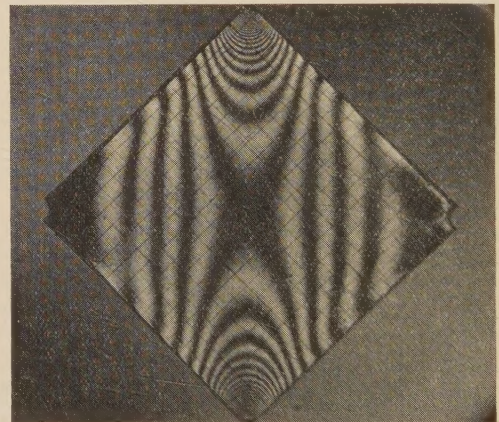


FIG. 16 ANNEALED SQUARE PLATE OF BAKELITE

The principal stresses were first determined along the diagonals, and for these lines Equations [8a] and [9a] are suitable. The notation is as follows: Referring to Fig. 19, Δx and Δy are the coordinates of any point C on the first curved isoclinic; $\Delta\phi$ is the difference between the parameters of the zero isoclinic and the first curved isoclinic. In the present case $\Delta\phi$ is 5 deg.

In the foregoing equations the directions of s_1 and s_2 are inter-related, s_2 being obtained from s_1 by a counterclockwise rotation of 90 deg.

The steps involved in the determination of P and Q will be best understood by a study of Tables 1 and 2. The mean values given in columns 6 and 12, Table 1, were taken from curves.

P was calculated first. The starting point was taken at A where P_0 is zero, Fig. 15, and integrated toward the origin O . For s_1 positive to the left, s_2 is positive downward. It follows that Δy in Equation [8a] is negative for the 5-deg. isoclinic. In the calculations Δs_1 was taken as 0.5 in., as the author was working from an image with a diagonal of 10 in. divided into twenty equal intervals. Having determined P and knowing $(P - Q)$ at the same points, it was a simple matter to find Q .

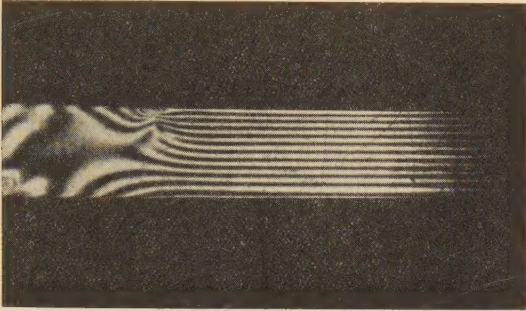


FIG. 17 ANNEALED BEAM OF BAKELITE

The stresses along the Y -axis were determined in a similar manner. The starting point was taken at O , at which point both principal stresses were now known. Using Equation [9a] and starting with Q_0 , values of Q were determined up to $0.82h$, which represents the limit of the experimental results. All the steps and the final results are given in Table 2.

The results thus far obtained are in terms of fringes. It is, however, a simple matter to convert these into pounds per square inch. The photograph of the comparison beam, Fig. 17, shows six fringes corresponding to a bending moment of 10.125 in.-lb. For a loading as shown in Fig. 20, the ordinary beam equation $S = Mv/I$ is applicable and gives $S_{\max.} = 1703.28$ lb. per sq. in., or 283.88 lb. per sq. in., for fringe No. 1.

By multiplying the fringe value of the principal stresses by 283.88 we obtain these stresses in pounds per square inch.

Verification of Results. The external load on the model was

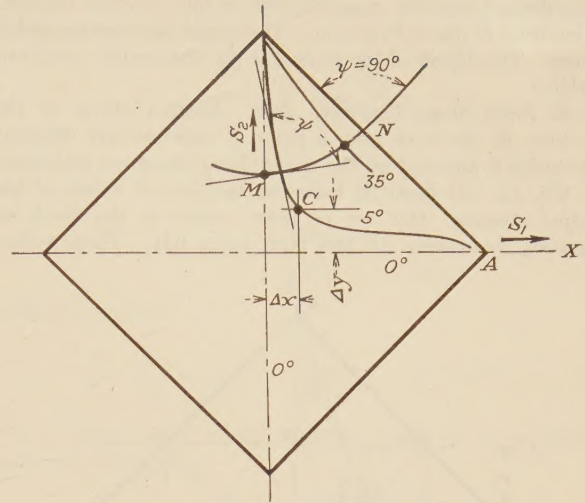


FIG. 19

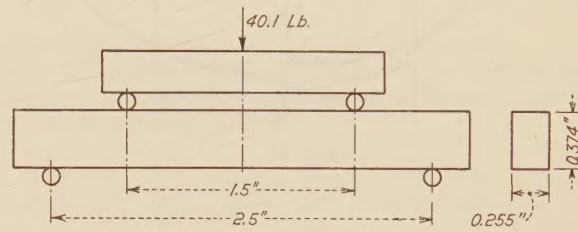


FIG. 20

179.66 lb. If, therefore, a section be made through the horizontal diagonal and one half of the block be considered as a free body, a static check of the work can be obtained. For equilibrium

$$\Sigma Y = 0$$

or

$$179.66 = \int_{-h}^h Q t dx \dots \dots \dots [10]$$

where $2h$ is the diagonal and t the thickness of the model. For $2h = 1.855$, $t = 0.255$, and the mean values of Q as given in Table 1 the right side of Equation [10] works out to be 191 lb., which gives a check within 5.7 per cent. A portion of the discrepancy is probably due to the small initial stresses at the edges of the comparison beam.

DETERMINATION OF PRINCIPAL STRESSES ON A GENERAL STRESS TRAJECTORY

To show the generality of the photoelastic method, the principal stresses were also computed along the stress trajectory MN , Fig. 15. An enlargement of the photograph shown in Fig. 16 was made and the stress trajectory MN superimposed upon it. The intersections of the fringes with this trajectory were noted and a curve plotted giving $(P - Q)$ as a function of MN . The intersections of the isoclinics with MN are marked by the small letters a, b, c, d, e, f , and g , Fig. 15.

To determine P and Q separately, Equation [8b] was used. $\Delta\phi$ in this case is the difference between the parameters of two successive isoclinics. Thus in going from N to g , $\Delta\phi$ is -10 deg., Fig. 15.

ψ for any point on the stress trajectory is the angle between the tangents to the stress trajectory and the intersecting isoclinic measured counterclockwise. These were determined from Fig. 15.

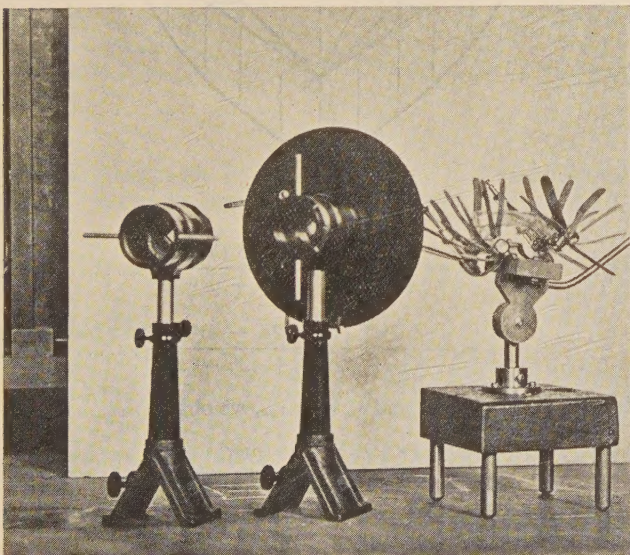


FIG. 18 SHOWING MERCURY-VAPOR LAMP AND CROSSED AHRENS PRISMS

The stress trajectory was followed in the direction of NM , starting from N where P_0 is zero. This made $\Delta\phi$ negative and P positive. The details of the work and the final results are given in Table 3.

Check From Stress Trajectory MN. Another check of the correctness of the work from a point of view entirely different from statics is afforded by the results from the stress trajectory MN , Fig. 15. At point M there are two sets of values of the principal stresses. One set of these values is the result of integrating first along AO and then along OM . These values

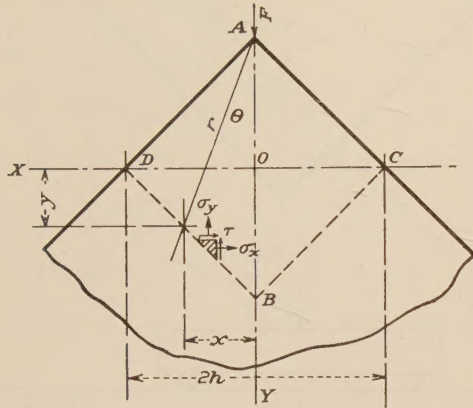


FIG. 21

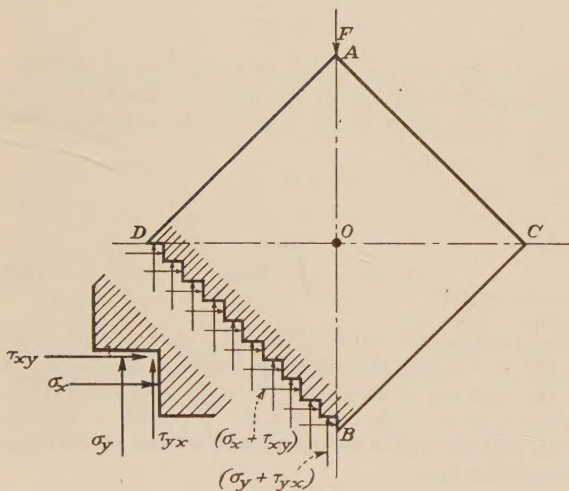


FIG. 22

in fringe order are: $P = 0.8020$, and $Q = -3.988$, see Table 2. The other set of values is the result of integrating along the path NM , which gave $P = 0.824$, and $Q = -3.976$ in fringe order, see Table 3.

The P 's agree within 2.8 per cent and the Q 's within 0.5 per cent. This agreement between the values of the principal stresses at a point obtained from different paths of integration constitutes the best proof of the correctness of the work.

SUMMARY OF APPROXIMATE THEORETICAL SOLUTION OF PROBLEM

The problem discussed is approached from the point of view of an infinite wedge, and the author assumes the stress function:⁶

⁶ It is shown in texts on elasticity that this function satisfies the boundary conditions and the continuity equation.

TABLE 3 COMPUTATIONS OF P AND Q ALONG STRESS TRAJECTORY MN

$P = P_0 + \int_{\phi_0}^{\phi} (P - Q) \cot \psi d\phi$. In going from N to M , $\Delta\phi$ is negative, and $P_0 = 0$. $\therefore P = \int_{\phi_0}^{\phi} (P - Q) \cot \psi d\phi$, a positive number

Deg.	$(P - Q)$	$\cot \psi$	$(P - Q) \times \cot \psi$	Mean $(P - Q) \cot \psi$	$(P - Q) \cot \psi \Delta\phi$	P in fringe order	Q in fringe order
0	4.80	0	0	-0.69	-0.0602	0.8240	-3.976
5	4.68	-0.231	-1.080	-1.270	-0.1110	0.7638	-3.916
10	4.35	-0.325	-1.415	-1.510	-0.1320	0.6528	-3.697
15	3.82	-0.404	-1.540	-1.590	-0.1390	0.5208	-2.309
20	3.15	-0.504	-1.580	-1.480	-0.1290	0.3818	-2.768
25	2.70	-0.466	-1.260	-1.200	-0.1050	0.2528	-2.450
30	2.42	-0.425	-1.030	-0.940	-0.0820	0.1478	-2.270
35	2.25	-0.335	-0.755	-0.400	-0.0658	0.0658	-2.180
45	2.00	0.0	0	...	...	0.0	-2.000

NOTE: $\Delta\phi = 5^\circ = 0.0873$ radian everywhere except at 45° , where $\Delta\phi = 10^\circ$ or 0.1646 rad.

$$\phi = Ar \theta \sin \theta \dots \dots \dots [11]$$

If from the infinite wedge we cut a square plate of a diagonal $2h$ as shown in Fig. 21, then the sides BD and BC would be subjected to stress components which in terms of cartesian coordinates and boundary conditions reduce to:

$$\sigma_x = -\frac{F}{(\pi + 2)} \frac{(y + h)(h - y)^2}{(h^2 + y^2)^2} \dots \dots \dots [12]$$

$$\sigma_y = -\frac{4F}{(\pi + 2)} \frac{(2h - x)^3}{[x^2 + (2h - x)^2]^2} \dots \dots \dots [13]$$

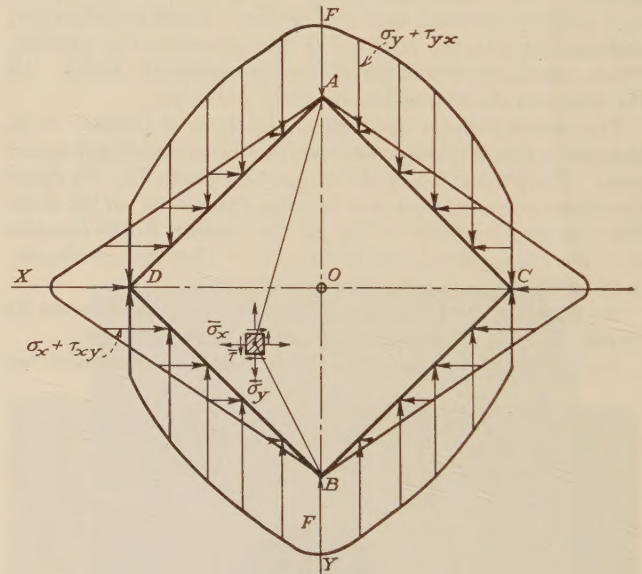


FIG. 23

$$\tau = -\frac{F}{(\pi + 2)} \frac{(y + h)^2 (h - y)}{(h^2 + y^2)^2} \dots \dots \dots [14]$$

$$= -\frac{4F}{(\pi + 2)} \frac{(2h - x)^2 x}{[x^2 + (2h - x)^2]^2} \dots \dots \dots [15]$$

Combining σ_x with τ_{xy} , and σ_y with τ_{yx} , we obtain

$$\sigma_x + \tau_{xy} = -\frac{F}{(\pi + 2)} \frac{2h(h^2 - y^2)}{(h^2 + y^2)^2} \dots \dots \dots [16]$$

and

$$\sigma_y + \tau_{yx} = -\frac{4F}{(\pi + 2)} \frac{(2h - x)^2 2h}{[x^2 + (2h - x)^2]^2} \dots \dots \dots [17]$$

Upon plotting curves for [16] and [17] we find that good approximations can be obtained with simpler and symmetrical functions: namely,

$$\sigma_x + \tau_{xy} = -0.195 \left(1 + \cos \frac{\pi x}{h} \right) \frac{F}{h} \dots \dots \dots [18]$$

$$\sigma_y + \tau_{yx} = - \left(0.564 - 0.1945 \frac{x^2}{h^2} - 0.1031 \cos \frac{\pi x}{h} - 0.0564 \cos \frac{2\pi x}{h} - 0.01556 \cos \frac{4\pi x}{h} \right) \frac{F}{h} \dots \dots \dots [19]$$

Returning now to the infinite wedge, a square plate $ABCD$ is cut from it whose free-body diagram is shown in Fig. 22. Suppose now that to the system of forces in Fig. 22 there is added a symmetrical system and boundary forces are obtained as shown in Fig. 23. The stress components for the combined system are:

$$\bar{\sigma}_x = - \frac{4Fx^2}{(\pi + 2)} \left\{ \frac{(h+y)}{[x^2 + (h+y)^2]^2} + \frac{(h-y)}{[x^2 + (h-y)^2]^2} \right\} \dots [20]$$

$$\bar{\sigma}_y = - \frac{4F}{(\pi + 2)} \left\{ \frac{(h+y)^3}{[x^2 + (h+y)^2]^2} + \frac{(h-y)^3}{[x^2 + (h-y)^2]^2} \right\} \dots [21]$$

and

$$\bar{\tau}_{xy} = \bar{\tau}_{yx} = \frac{4Fx}{(\pi + 2)} \left\{ \frac{(h-y)^2}{[x^2 + (h-y)^2]^2} - \frac{(h+y)^2}{[x^2 + (h+y)^2]^2} \right\} [22]$$

If to $\bar{\sigma}_x$, $\bar{\sigma}_y$, and $\bar{\tau}$ are added the stress components due to a system of boundary forces equal and opposite to those shown in Fig. 23 the required stress distribution will be obtained, that is, the stress system in a square plate subjected to diagonal compression. This will be done by the use of Professor Timoshenko's Approximate Strain-Energy Method.⁷

The required stress function is assumed in the form of a series, thus

$$\Phi = \Phi_0 + \alpha_1 \phi_1 + \alpha_2 \phi_2 + \dots \dots \dots \alpha_n \phi_n \dots \dots [23]$$

where Φ_0 is a function so chosen that its corresponding stress system satisfies the boundary conditions, $\phi_1, \phi_2, \dots \dots \phi_n$ are functions whose corresponding stress systems vanish on the boundary, and $\alpha_1, \alpha_2, \dots \dots \alpha_n$ are constant coefficients determined by the conditions that the strain energy at equilibrium must be a minimum.

We shall limit ourselves to the first two terms of the series [23], i.e.,

$$\Phi = \Phi_0 + \alpha_1 \phi_1 \dots \dots \dots [24]$$

The expression for the strain-energy V , assuming Poisson's ratio $\mu = 0$, and the modulus of elasticity equal E is

$$V = \frac{1}{2E} \iint (\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2) \times dxdy \dots \dots [25]$$

and for equilibrium

$$\frac{\partial V}{\partial \alpha_1} = 0 = \iint \left(\sigma_x \frac{\partial \sigma_x}{\partial \alpha_1} + \sigma_y \frac{\partial \sigma_y}{\partial \alpha_1} + 2\tau_{xy} \frac{\partial \tau_{xy}}{\partial \alpha_1} \right) dxdy [26]$$

It is proved in the theory of elasticity that in terms of a stress function F the stress components are:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad \text{and} \quad \tau = - \frac{\partial^2 F}{\partial x \partial y}$$

It follows that the stress components in terms of our approximate stress function Φ are:

$$\sigma_x = \frac{\partial^2 \Phi_0}{\partial y^2} + \alpha_1 \frac{\partial^2 \phi_1}{\partial y^2} \dots \dots \dots [27]$$

$$\sigma_y = \frac{\partial^2 \Phi_0}{\partial x^2} + \alpha_1 \frac{\partial^2 \phi_1}{\partial x^2} \dots \dots \dots [28]$$

$$\tau = - \left(\frac{\partial^2 \Phi_0}{\partial x \partial y} + \alpha_1 \frac{\partial^2 \phi_1}{\partial x \partial y} \right) \dots \dots \dots [29]$$

It will be found that

$$\phi_1 = \frac{1}{16} [(x+y)^2 - h^2]^2 [(y-x)^2 - h^2]^2 \dots \dots [30]$$

gives a stress system which vanishes on the given boundaries whose equations are

$$x + y = \pm h \dots \dots \dots [31]$$

$$y - x = \pm h \dots \dots \dots [32]$$

and therefore satisfies the conditions imposed upon it.

Upon substitution of Equations [27], [28], and [29] and their respective derivatives with respect to α_1 in Equation [26] and upon collecting terms in α_1 we get

$$\iint \left\{ \frac{\partial^2 \Phi_0}{\partial y^2} \times \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \Phi_0}{\partial x^2} \times \frac{\partial^2 \phi_1}{\partial x^2} + 2 \frac{\partial^2 \Phi_0}{\partial x \partial y} \frac{\partial^2 \phi_1}{\partial x \partial y} + \alpha_1 \left[\left(\frac{\partial^2 \phi_1}{\partial y^2} \right)^2 + \left(\frac{\partial^2 \phi_1}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \phi_1}{\partial x \partial y} \right)^2 \right] \right\} dxdy = 0 [33]$$

Letting X and Y stand for $-(\sigma_x + \tau_{xy})$ and $-(\sigma_y + \tau_{yx})$, respectively, we remove first the boundary forces defined by Y . In this case Φ_0 is a function of X only. It will also be found that within the given limits of integration

$$\int_0^h \int_0^{h-y} \frac{\partial^2 \phi_1}{\partial x^2} dxdy = \int_0^h \int_0^{h-y} \frac{\partial^2 \phi_1}{\partial y^2} dxdy \dots [34]$$

Taking these into consideration, Equation [33] reduces to

$$2\alpha_1 \int_0^h \int_0^{h-y} \left[\left(\frac{\partial^2 \phi_1}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \phi_1}{\partial x \partial y} \right)^2 \right] \times dxdy = - \int_0^h \int_0^{h-y} \frac{\partial^2 \Phi_0}{\partial x^2} \frac{\partial^2 \phi_1}{\partial x^2} dxdy \dots \dots \dots [35]$$

It is from this equation that we determine α_1 . The left side of Equation [35] contains about eighty integrals of the type

$$\iint x^m y^n dxdy$$

These have been reduced to only sixteen by observing that within the given limits of integration

$$\iint x^m y^n dxdy = \iint x^n y^m dxdy \dots \dots \dots [36]$$

With this simplification

$$2 \int_0^h \int_0^{h-y} \left[\left(\frac{\partial^2 \phi_1}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \phi_1}{\partial x \partial y} \right)^2 \right] dxdy = \int_0^h \int_0^{h-y} (25x^{12} - 78x^{10}y^2 - 106h^2x^{10} + 222h^2x^8y^2 + 87x^8y^4 + 175h^4x^8 - 116h^2x^6y^4 - 220h^4x^6y^2 - 34x^6y^6 - 140h^6x^6 + 109h^4x^4y^4 + 55h^8x^4 + 76h^6x^4y^2 + 5h^8x^2y^2 - 10h^{10}x^2 + 0.5h^{12}) dxdy = 0.1044997h^{14} [37]$$

⁷ See his article in *Phil. Mag.*, vol. XLVII, June, 1924.

The right side of Equation [35] is

$$-\int_0^h \int_0^{h-y} \frac{\partial^2 \Phi_0}{\partial x^2} \frac{\partial^2 \phi_1}{\partial x^2} dx dy = -\int_0^h \int_0^{h-y} (3.5x^6 + 4.5x^2y^4 + 4.5h^4x^2 - 7.5x^4y^2 - 7.5h^2x^4 - 0.5y^6 + 3h^2x^2y^2 + 0.5h^2y^4 + 0.5h^4y^2 - 0.5h^6) \times \left(0.564 - 0.1945 \frac{x^2}{h^2} - 0.1031 \cos \frac{\pi x}{h} - 0.0564 \cos \frac{2\pi x}{h} - 0.01556 \cos \frac{4\pi x}{h} \right) \frac{F}{h} dx dy = -0.0070967 Fh^7. [38]$$

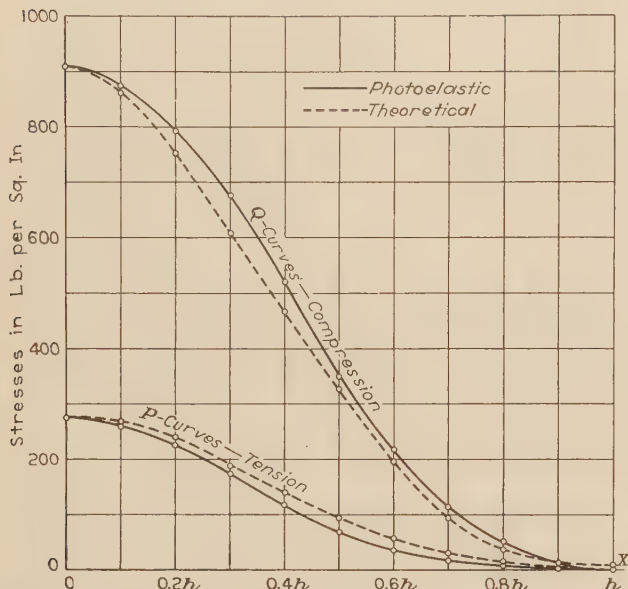


FIG. 24 COMPARATIVE CURVES OF THEORETICAL AND PHOTOELASTIC VALUES OF P AND Q ALONG OX

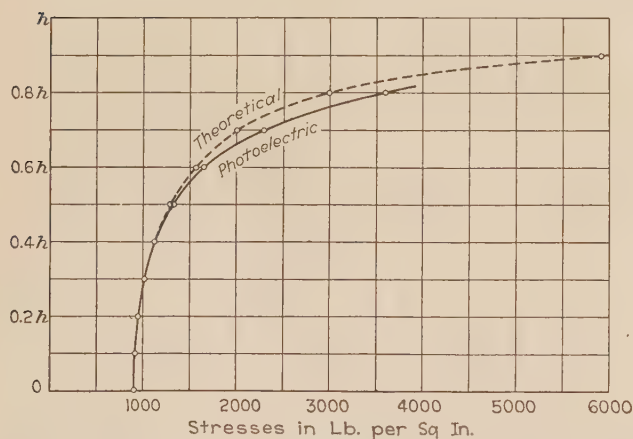


FIG. 25 COMPARATIVE CURVES OF THEORETICAL AND PHOTOELASTIC VALUES OF P ALONG OY

Solving for α_1 .

$$\alpha_1 = -\frac{0.0070967 Fh^7}{0.1044997 h^{14}} = -0.06788 \frac{F}{h^7} \dots [39]$$

The approximate stress function for the Y boundary forces thus becomes

$$\Phi = \Phi_0 - 0.06788 \frac{F}{h^7} \times \frac{1}{16} [(x+y)^2 - h^2]^2 \times [(y-x)^2 - h^2]^2 \dots [40]$$

In precisely the same way the approximate stress function for the X boundary forces is found to be

$$\psi = \psi_0 + 0.114249 \frac{F}{h^7} \times \frac{1}{16} [(x+y)^2 - h^2]^2 \times [(y-x)^2 - h^2]^2 [41]$$

where ψ and ψ_0 replace Φ and Φ_0 , respectively. Upon combining the two stress systems Φ and ψ and designating the resultant stress components by P_0 , Q_0 , and S_0 , we obtain

$$P_0 = \frac{\partial^2 \psi_0}{\partial y^2} + 0.04637 \frac{F}{h^7} \frac{\partial \phi_1}{\partial y^2} \dots [42]$$

and

$$Q_0 = \frac{\partial^2 \Phi_0}{\partial x^2} + 0.04637 \frac{F}{h^7} \frac{\partial \phi_1}{\partial x^2} \dots [43]$$

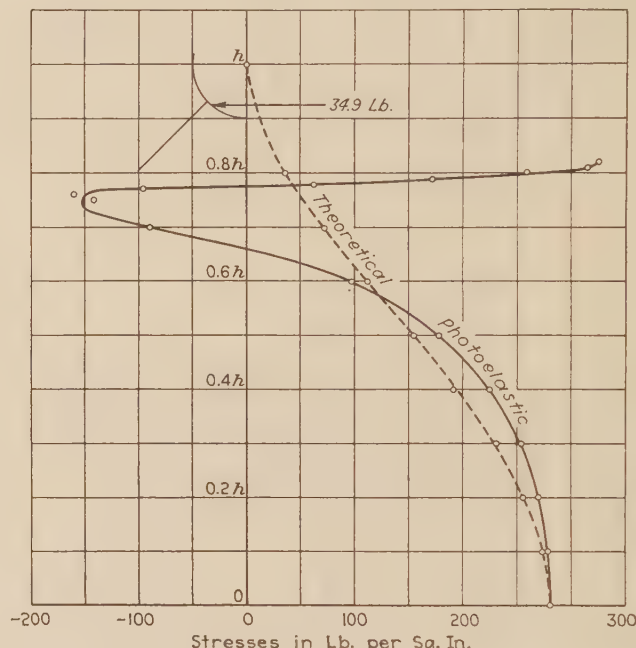


FIG. 26 COMPARATIVE CURVES OF THEORETICAL AND PHOTOELASTIC VALUES OF Q ALONG OY

Denoting the resultant stress components in the square block by P , Q , and S , respectively, we obtain

$$P = P_0 + \bar{\sigma}_x \dots [44]$$

$$Q = Q_0 + \bar{\sigma}_y \dots [45]$$

Table 4 gives the values of P and Q for the axes of symmetry; $= x/h$ or y/h .

TABLE 4 GIVING P AND Q ALONG X - AND Y -AXES

r	X -axis		Y -axis	
	P in $\frac{F}{h}$	Q in $\frac{F}{h}$	P in $\frac{F}{h}$	Q in $\frac{F}{h}$
0.0	0.3668	-1.1900	0.3668	-1.1900
0.1	0.3520	-1.1300	0.3593	-1.2040
0.2	0.3100	-0.9850	0.3379	-1.2530
0.3	0.2510	-0.8000	0.3025	-1.3420
0.4	0.1860	-0.6130	0.2573	-1.4800
0.5	0.1250	-0.4300	0.2048	-1.7000
0.6	0.0741	-0.2600	0.1492	-2.0540
0.7	0.0380	-0.1270	0.0950	-2.6700
0.8	0.0151	-0.0498	0.0476	-3.9360
0.9	0.0045	-0.0170	0.0134	-7.7930
1.0	0.0010	0.0116	0.0	∞

DISCUSSION OF RESULTS

The extent to which the theoretical results substantiate the

experimental ones is graphically represented by the curves of Figs. 24-26, inclusive. The only marked divergence is shown by the curve of Fig. 26 beyond $0.6h$. The explanation for this discrepancy is found in the approximate character of the stress functions Φ and ψ . It is felt that if more terms of the series [23] were taken this divergence would be minimized. All the other curves substantiate the photoelastic results. At the center and the extremities of the horizontal diagonal the agreement is almost perfect. It is therefore considered that the results from the photoelastic investigation represent a correct picture of the stress distribution in the square block.

Referring now to this stress distribution it is observed first of all that fairly large tensile stresses are produced as a result of the diagonal compression. From an engineering point of view this is perhaps the most interesting aspect of the problem. The large tensile stress could hardly have been predicted on the basis of the ordinary theories of mechanics. At the center the maximum tension is 278.6 lb. per sq. in., or nearly one-third of the corresponding compressive stress. The maximum value of the tension, however, is at least 350 lb. per sq. in., this being the value obtained on the Y -axis at $0.82h$. Square blocks weak in tension and subjected to diagonal compression should therefore show signs of failure at points on the loaded diagonal about $h/5$ from the corners.

It may also be worth noting that the P -stresses are all tensile along what might be called the horizontal stress trajectories, but are of variable signs on the vertical stress trajectories. Thus on the Y -axis the P is positive from 0 to $0.655h$, negative from this point to $0.775h$, and again positive from there on to at least $0.82h$. No computations were made beyond this point due to the inaccuracies in the measurements of the very small values of $\Delta\epsilon$. It is of interest that $(P - Q)$ in fringe order could be distinctly read even at $0.9h$.

The P -curve cannot rise much beyond the value of 350 lb. per sq. in. without having to make a sudden return and cross the axis again to assume compressive values. Assuming that the P -curve stays positive to $0.9h$, and that the stress distribution in the tip beyond $0.9h$ is essentially that of an infinite wedge, we get a static check on the loaded diagonal from horizontal summation of the forces, first, however, cutting out the loaded tips up to a radius of $0.1h$, and replacing them by the horizontal thrusts which they exert, and which for our loads were found to be 34.9 lb.

$$\Sigma X = 0, \text{ or } 34.9 = \int_0^h P t dy \dots \dots \dots [46]$$

Taking $2h$ again as 1.855 in., t as 0.255 in., and the mean values of P from Table 2, we get for the right side of Equation [46] 33.96 lb., or a static check within 2.7 per cent.

PHOTOELASTIC EQUIPMENT

The main equipment of a photoelastic laboratory consists of

- 1 A polariscope
- 2 A compensator
- 3 A loading frame
- 4 A camera
- 5 A monochromatic lamp
- 6 An electric furnace, and
- 7 A special table for drawing isoclinics.

Fig. 10 shows a photograph of the polariscope used in the Photoelastic Laboratory of the Mechanics Department at the University of Michigan. It is imported from the Winkel-Zeiss firm at Göttingen. It uses Ahren prisms instead of Nicols, and gives satisfactory results.

Fig. 27 shows a photograph of a compensator developed at the laboratory. The carriage of the compensator was designed by Dr. Donnell, formerly of the Mechanics Department of the University of Michigan. It is capable of a vertical, horizontal, and rotary motion. The spring, the beam compensator (Fig. 28), the setting-up fixture, and other features were designed by the author. Considerable difficulty was encountered in the design of a spring which would be free from energy losses. The adopted design gives a linear relation between load and deflection, and is the same during loading and unloading. In the fringe method the compensator is not used to determine $(P - Q)$. It is used, however, to determine the sign of the stress.

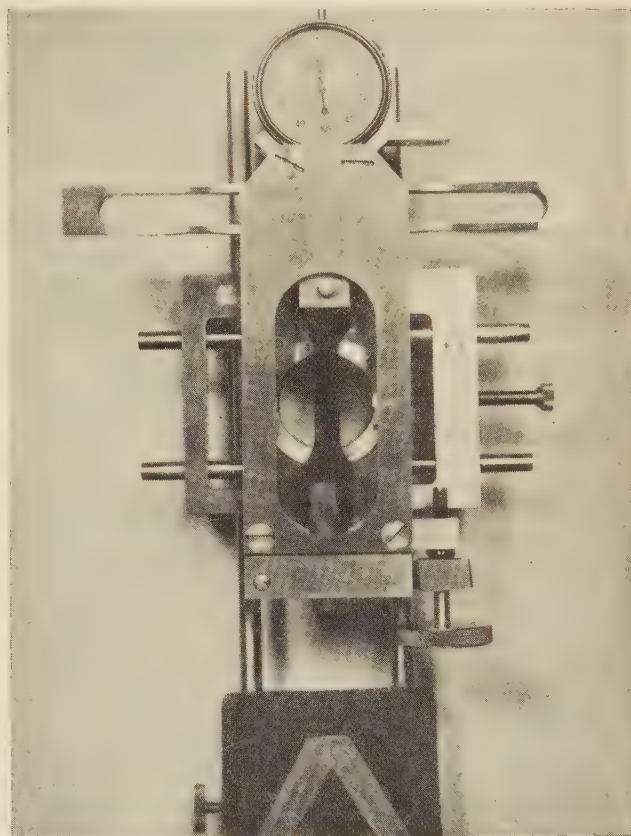


FIG. 27 TENSION COMPENSATOR

It also makes an excellent tension machine, and the one shown could produce a tension of 150 lb., the accuracy of the reading being $1/4$ lb. The spring constant is 1615 lb.

The Loading Frame. The frame shown in Fig. 29 can produce tension, compression, and bending. The central frame is capable of a vertical and a horizontal motion. It can accommodate many sizes of models and has means to produce perfect alignment of the supporting pins. When producing bending provision is made for self-centering, which automatically gives equal moment arms and loads. By means of a small eccentric operated by a wooden handle visible in the picture, the load can be quickly removed from the model. The springs reduce impact forces and absorb vibrations. The main lever rests against a knife edge.

Camera. An ordinary Eastman 5×7 -in. camera was used. The lens was removed, the front of the camera opened, and the image projected directly upon the film. To retain the use of the shutter it was mounted on a special pedestal and placed close to the analyzer. In selecting a camera for the fringe

method special attention should be paid to the provisions for adjusting the focal plane, as it is of utmost importance to secure good focusing of every part of the image.

Lamp. For the fringe method a monochromatic lamp must be had. The one used was a mercury-vapor lamp, shown in Fig. 18. With the aid of a No. 77 Wratten filter, the yellow rays were stopped and a fairly pure green light obtained. Orthochromatic Eastman films were found suitable for the author's purpose.

Furnace and Isoclinic Table. The annealing work was done in a small electric furnace working at $12\frac{1}{2}$ amperes and 220 volts. Fig. 30 shows a three-in-one piece of equipment. The main function of this table was to provide a vertical glass drawing board on which to draw the isoclinics. By placing an enameled screen in the grooves provided in the uprights, a projecting screen is obtained which can be moved to any desired position. Also, upon removing the special frame used for the glass plate and attaching a wooden shelf through bolts in the back of the table, a satisfactory support is obtained for the camera.

CONCLUSION

Aside from the results obtained for the stress distribution in square plates under diagonal compression which may have a practical bearing in the design and understanding of the behavior of double knife edges, this investigation affords an opportunity to compare the several methods in photoelasticity and to contrast the photoelastic method with the theoretical. It is the author's opinion that the fringe method, whenever it can be used, is superior to compensating or color matching both in accuracy and simplicity, and that with the successful annealing

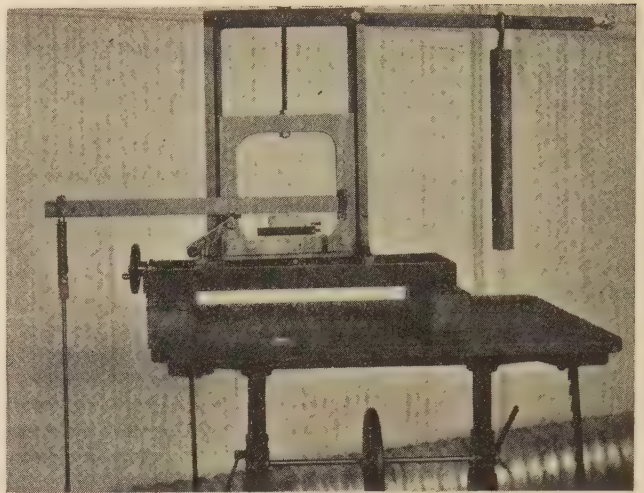


FIG. 29 LOADING FRAME SHOWING ARRANGEMENT FOR BENDING
(See Fig. 10 for compression.)



FIG. 30 COMBINED PROJECTING SCREEN, VERTICAL DRAWING BOARD, AND CAMERA SUPPORT

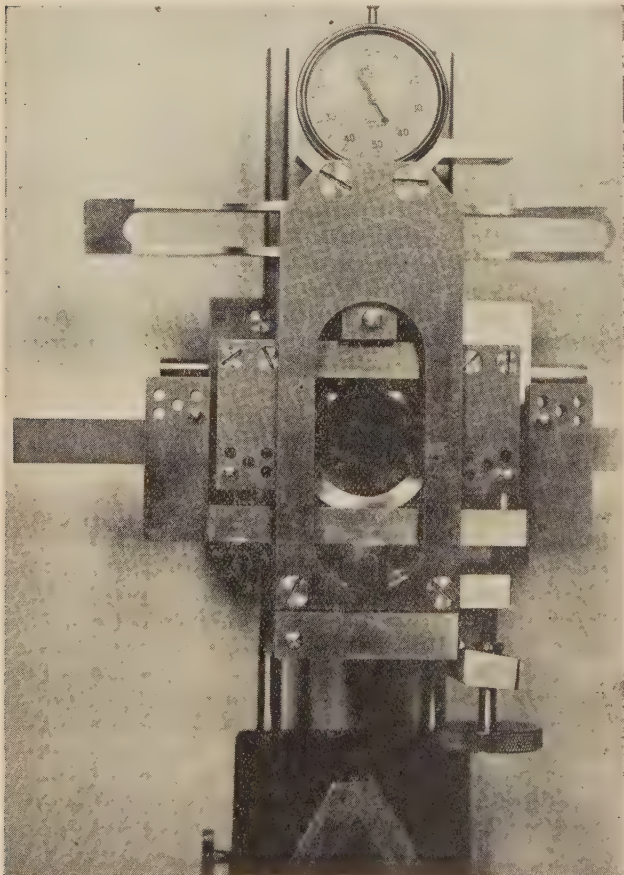


FIG. 28 BEAM COMPENSATOR

of bakelite a suitable material is made available for this method.

That the photoelastic results are reliable and agree with those from the mathematical theory of elasticity has been proved long before this. It is only desired to call attention to the vast difference in time and labor between the two methods. The summary of the approximate theoretical solution submitted fails to convey the amount of work necessary to get the results. Special tables of integrals have to be constructed, and many laborious and tedious computations have to be made which increase the danger of errors. It has been the author's experience that the photoelastic method takes about as many weeks as the theoretical method takes months.

The prevailing impression seems to be that the cost of a photoelastic laboratory is too high for the average engineering college or industrial organization. It may therefore be of interest to note that all the equipment described in this paper cost less than \$1500, and it is believed that \$2000 is sufficient to equip a moderate but efficient laboratory for quantitative work.

The main problem of this paper was suggested by Professor Timoshenko, who has kept in constant touch with this investi-

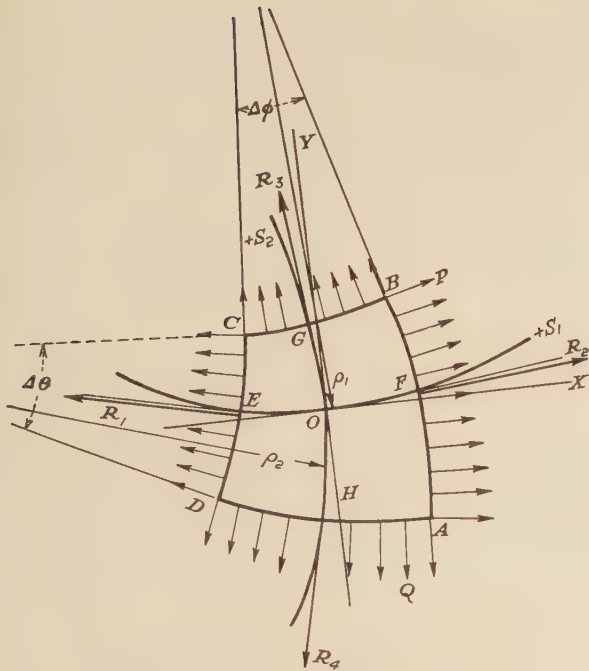


FIG. 31

gation and to whom the author is indebted for valuable advice. He also wishes to thank Dr. Tuzi for the sample of phenolite used in the investigation, and the Research Department of the Westinghouse Electric & Manufacturing Company for the bakelite which they were good enough to supply for the research.

Appendix No. 1

DERIVATION OF FUNDAMENTAL EQUATIONS FOR THE METHOD OF GRAPHICAL INTEGRATION

Let s_1 and s_2 denote two stress trajectories meeting at point O , Fig. 31.

Let s_1 be measured positive to the right of O and s_2 upward.

Let ρ_1 and ρ_2 be radii of curvature of s_1 and s_2 , respectively, being measured positive if the tangent to the curve rotates counterclockwise as s_1 and s_2 increase.

In the element of Fig. 31 $OE = OF = \Delta s_1$, and $OG = OH = \Delta s_2$; also the boundary consists of arcs of the stress trajectories passing through points H , F , and G , and E .

Let P and Q represent average boundary stresses, and let R_1 , R_2 , R_3 , and R_4 represent the resultant forces on the four sides of the element.

Lastly, let P_0 and Q_0 represent the resultant forces on GOH and EOF , respectively.

If $\Delta\phi$ is a small angle, the resultants R_1 and R_2 may be treated as tangents to EOF at E and F , respectively. Similarly R_3 and R_4 may be taken as tangent to GOH at G and H , respectively, the errors being infinitesimals of higher order. It follows that R_1 and R_2 make angles with the X -axis equal to $\Delta\phi/2$.

The Y -component of R_1 is $R_1 \sin \frac{\Delta\phi}{2}$

The Y -component of R_2 is $R_2 \sin \frac{\Delta\phi}{2}$

and the sum, which will be designated by R_y , is

$$R_y = (R_1 + R_2) \sin \frac{\Delta\phi}{2} = (R_1 + R_2) \frac{\Delta\phi}{2} \dots \dots [47]$$

$\Delta\phi$ being a small angle.

For ρ_1 a positive radius of curvature R_y is positive that is upward, as can be seen from the figure. For equilibrium it is necessary that

$$\Sigma Y = 0 \dots \dots \dots [48]$$

Hence

$$(R_4 - R_3) = R_y \dots \dots \dots [49]$$

$$\therefore (R_4 - R_3) = (R_1 + R_2) \frac{\Delta\phi}{2} \dots \dots \dots [50]$$

Now

$$R_1 = \left(P_0 - \frac{\partial P_0}{\partial s_1} \Delta s_1 \right) \dots \dots \dots [51]$$

and

$$R_2 = \left(P_0 + \frac{\partial P_0}{\partial s_1} \Delta s_1 \right) \dots \dots \dots [52]$$

hence

$$R_1 + R_2 = 2P_0 \dots \dots \dots [53]$$

and [47] becomes

$$(R_1 + R_2) \frac{\Delta\phi}{2} = 2P_0 \frac{\Delta\phi}{2} = P_0 \Delta\phi \dots \dots \dots [54]$$

Also

$$R_3 = Q_0 + \frac{\partial Q_0}{\partial s_2} \Delta s_2 \dots \dots \dots [55]$$

and

$$R_4 = Q_0 - \frac{\partial Q_0}{\partial s_2} \Delta s_2 \dots \dots \dots [56]$$

$$\therefore R_4 - R_3 = -2 \frac{\partial Q_0}{\partial s_2} \Delta s_2 \dots \dots \dots [57]$$

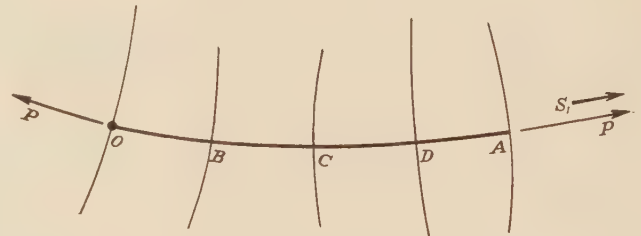


FIG. 32

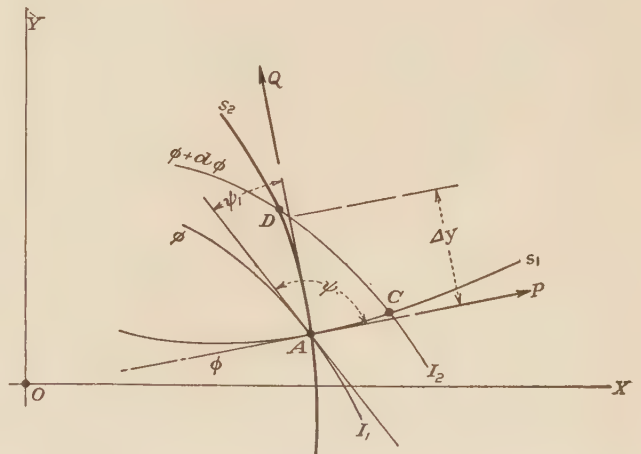


FIG. 33

Combining with [53],

$$-2 \frac{\partial Q_0}{\partial s_2} \Delta s_2 = P_0 \Delta \phi \dots \dots \dots [58]$$

or

$$\frac{\partial Q_0}{\partial s_2} \Delta s_2 = -\frac{P_0}{2} \Delta \phi \dots \dots \dots [59]$$

But as a first approximation

$$Q_0 = 2 \Delta s_1 Q \dots \dots \dots [60]$$

and

$$P_0 = 2 \Delta s_2 P \dots \dots \dots [61]$$

On substitution in [59],

$$2 \frac{\partial}{\partial s_2} (\Delta s_1 Q) \Delta s_2 = -2 \Delta s_2 P \frac{\Delta \phi}{2} \dots \dots \dots [62]$$

but

$$\Delta s_1 = \rho_1 \frac{\Delta \phi}{2}$$

hence [62] becomes

$$2 \frac{\partial}{\partial s_2} \left[\left(\rho_1 \frac{\Delta \phi}{2} \right) Q \right] \Delta s_2 = -\Delta s_2 P \Delta \phi \dots \dots \dots [63]$$

But $\Delta \phi$ may be taken as a constant, so that [63] becomes

$$\frac{\partial}{\partial s_2} (\rho_1 Q) \Delta s_2 \Delta \phi = -\Delta s_2 P \Delta \phi \dots \dots \dots [64]$$

or

$$\frac{\partial}{\partial s_2} (\rho_1 Q) = -P \dots \dots \dots [65]$$

$$\therefore \rho_1 \frac{\partial Q}{\partial s_2} + Q \frac{\partial \rho_1}{\partial s_2} = -P \dots \dots \dots [66]$$

In the limit $\frac{\partial \rho_1}{\partial s_2} = -1$, since $\Delta \rho_1 = -\Delta s_2$ and we have $\rho_1 \frac{\partial Q}{\partial s_2} - Q = -P$

$$\therefore \frac{\partial Q}{\partial s_2} + \frac{P - Q}{\rho_1} = 0 \dots \dots \dots [67]$$

Similarly,

$$\frac{\partial P}{\partial s_1} + \frac{P - Q}{\rho_2} = 0 \dots \dots \dots [68]$$

It will be observed that Equations [67] and [68] do not contain the elastic constants. Therefore they hold beyond the elastic limit.

If we start from any point O on the stress trajectory of P and follow it to any point A we obtain upon integration

$$\int_{P_0}^{P_A} dP = - \int_0^A \frac{(P - Q)}{\rho_2} ds_1 \dots \dots \dots [69]$$

or

$$P_A - P_0 = - \int_0^A \frac{(P - Q)}{\rho_2} ds_1 \dots \dots \dots [70]$$

Hence

$$P_A = P_0 - \int_0^A \frac{(P - Q)}{\rho_2} ds_1 \dots \dots \dots [71]$$

It must be remembered that in this derivation ρ_1 and ρ_2 were assumed positive if the rotation of the tangent to the curve is counterclockwise as the curves s_1 and s_2 increase and that s_2

and s_1 are interrelated, s_2 being obtained from s_1 by counterclockwise rotation of 90 deg. Thus in Fig. 32 if O is the origin and s_1 is measured positive to the right, ρ_1 is positive at all points between O and A , and ρ_2 is positive at O , B , and A , and negative at C and D .

In Fig. 32 if A is taken as an origin and s_1 is measured positive toward O , then ρ_1 is negative at all points; ρ_2 is negative at A , B , and O and is positive at D and C .

Professor Filon⁸ gives two methods to evaluate the integrals on the right-hand sides of Equations [71] and [72]:

$$P_A = P_0 - \int_0^A \frac{P - Q}{\rho_2} ds_1 \dots \dots \dots [71]$$

$$Q_A = Q_0 - \int_0^A \frac{P - Q}{\rho_1} ds_2 \dots \dots \dots [72]$$

Let point A be a point in a body under plane stress, Fig. 33. Through this point there are two stress trajectories s_1 and s_2 and an isoclinic I_1 of parameter ϕ , where ϕ is the angle which the tangent to s_1 at A makes with the X -axis.

Let I_2 be a nearby isoclinic of parameter $\phi + d\phi$, intersecting s_1 at C and s_2 at D . Then

$$AD = \rho_2 d\phi; \text{ and } \frac{1}{\rho_2} = \frac{d\phi}{AD} \dots \dots \dots [73]$$

Also

$$ds_1 = AC$$

hence

$$\frac{1}{\rho_2} ds_1 = d\phi \frac{AC}{AD} \dots \dots \dots [74]$$

If ψ be the angle through which the stress trajectory of P has to be rotated (counterclockwise) in order to bring it upon the isoclinic I_1 , then

$$\frac{AD}{AC} = -\tan \psi \dots \dots \dots [75]$$

the negative sign being necessary because $\tan \psi$ is negative and AD/AC is positive.

Upon substitution in [74], we have

$$\frac{ds_1}{\rho_2} = \frac{-d\phi}{\tan \psi} = -\cot \psi d\phi \dots \dots \dots [76]$$

Substituting in [71]

$$P_A = P_0 + \int_{\phi_0}^{\phi} (P - Q) \cot \psi d\phi \dots \dots \dots [77]$$

Also

$$\rho_1 d\phi = AC, \text{ and } \frac{1}{\rho_1} = \frac{d\phi}{AC}$$

$$\therefore \frac{ds_2}{\rho_1} = d\phi \frac{AD}{AC} = d\phi \cot \psi_1 \dots \dots \dots [78]$$

Upon substitution in [72],

$$Q_A = Q_0 - \int_{\phi_0}^{\phi} (P - Q) \cot \psi_1 d\phi \dots \dots \dots [79]$$

The equations just discussed give good results when the angles ψ are large. For small values of ψ , Equations [77] and [79]

⁸ L. N. G. Filon, "On the Graphical Determination of Stress From Photo-Elastic Observation," *Engineering*, Oct., 1923, p. 511.

become inaccurate as a small error in ψ makes a large error in $\cot \psi$. In such case the intervals between successive isoclinics become large, and this adds to the inaccuracy of the method.

Professor Filon therefore gives another treatment for such cases. Let Δy , Fig. 33, be the intercept measured perpendicular to the path of integration between two near isoclinics in the neighborhood of the point considered whose parameters differ by $\Delta\phi$. Then

$$\frac{1}{\rho^2} = \frac{\Delta\phi}{\Delta y} \text{ (approximately) } \dots\dots\dots [80]$$

The isoclinics are usually drawn for constant differences of say, 5 or 10 deg., so that $\Delta\phi$ is a constant.

Substituting this in [67] and [68],

$$P = P_0 - \Delta\phi \int_0^A \frac{(P - Q)}{\Delta y} ds_1 \dots\dots\dots [81]$$

which again enables us by graphical integration to determine P and Q .

This last method is especially useful in cases of symmetry, in which the straight lines of symmetry are also isoclinics. Δy is then the intercept between this line and the nearest isoclinic, measured perpendicular to the path of integration.

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